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A DIFFERENTIAL GAME OF QUALITY FOR TWO GROUPS OF OBJECTS†

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A bilinear differential game of quality for two groups of objects with several terminal surfaces is considered. A discrete-approximation method is proposed, which generates the solution of the differential game as the limit of the solutions of multistep approximating games. The effect of a coordinated and an uncoordinated choice of strategies by the objects in a group on the outcome of the conflict is considered.

1. STATEMENT OF THE PROBLEM

THE MOTION of two groups of conflict-controlled objects $P_1, \ldots, P_M$ (group A) and $L_1, \ldots, L_K$ (group B), where M and K are natural numbers, is defined by the following system of differential equations

$$x_i^{\cdot}(t) = -x_i(t) + (E - U_i(t)) B_i^1 x_i(t) - \sum_{m=1}^{K} S_{im}^2 V_m(t) B_m^2 y_m(t)$$

$$y_j^{\cdot}(t) = -y_j(t) + (E - V_j(t)) B_j^2 y_j(t) - \sum_{m=1}^{M} S_{jm}^1 U_m(t) B_m^1 x_m(t)$$

$$x_i(t) \in R^n, \quad x_i(0) \in \operatorname{int} R_+^n \ (i = 1, \ldots, M), \ y_j(t) \in R^n,$$
$$y_j(0) \in \operatorname{int} R_+^n$$
$$(j = 1, \ldots, K), u_i^l(t) \in [0, 1], v_j^l(t) \in [0, 1] \ (l = 1, \ldots, n), \ u_i(t) =$$
$$=(u_i^1(t), \ldots, u_i^n(t)), \ v_j(t) = (v_j^1(t), \ldots, v_j^n(t))$$

Here $x_i(t)$ and $y_j(t)$ are the states of the objects P_i, L_j; E, B_i^1, B_j^2, S_{ij}^1, S_{ji}^2, $U_i(t)$, $V_j(t)$ are $n \times n$ matrices with positive elements, E is the identity matrix, $V_i(t)$ and $V_j(t)$ are diagonal elements with the elements $u_i^l(t)$, $v_j^l(t)$; $u_i(t)$, $v_j(t)$ are the values of the strategies of objects P_i, L_j at time t, $0 \leqslant t < +\infty$.

The game terminates when the following conditions are satisfied

$$(x_1(t), \ldots, x_M(t), y_1(t), \ldots, y_K(t)) \in \Big(\bigcap_{i=1}^{Mn} S_i\Big) \cap \Big(\bigcup_{j=1}^{Kn} T_j\Big) \tag{1.1}$$

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$$(x_1(t), \ldots, x_M(t), y_1(t), \ldots, y_K(t)) \in \left(\bigcap_{j=1}^{Kn} T_j^*\right) \cap \left(\bigcup_{i=1}^{Mn} S_i^*\right) \tag{1.2}$$

$$S_i = \{(x_1, \ldots, x_M, y_1, \ldots, y_K) : (x_1, \ldots, x_M, y_1, \ldots, y_K) \in R_+^{n(K+M)} x^i > 0\}$$

$$S_i^* = \{(x_1, \ldots, x_M, y_1, \ldots, y_K) : (x_1, \ldots, x_M, y_1, \ldots, y_K) \in R_+^{n(K+M)} x^i = 0\}$$

x^i is a component of the vector $(x_1, \ldots, x_M)$ $(i = 1, \ldots, Mn)$,

$$T_j = \{(x_1, \ldots, x_M, y_1, \ldots, y_K) : (x_1, \ldots, x_M, y_1, \ldots, y_K) \in R_+^{n(K+M)} y^j = 0\}$$

$$T_j^* = \{(x_1, \ldots, x_M, y_1, \ldots, y_K) : (x_1, \ldots, x_M, y_1, \ldots, y_K) \in R_+^{n(K+M)},\ y^j > 0\}$$

y_j is a component of the vector $(y_1, \ldots, j_K)$ $(j = 1, \ldots, Kn)$.

The information available to the objects essentially influences the outcome of the conflict and the structure of their optimal strategies [1–3]. We therefore assume that the objects in the two groups use positional strategies. Thus, as in [4], the original game defines two problems: the problem from the point of view of group A and the problem from the point of view of group B. Group A or B applies positional strategies to each problem and these strategies may be chosen in different ways by the objects. We consider two extreme cases:

1. The objects in a group coordinate the choice of their strategies, acting as a single object.
2. The objects in a group choose their own strategies without any coordination with other objects in the group.

2. COORDINATED CHOICE OF STRATEGIES BY THE OBJECTS IN A GROUP

Denote by z, x and y the vectors $(x_1, \ldots, x_M, y_1, \ldots, y_K)$, $(x_1, \ldots, x_M)$ $(y_1, \ldots, y_K)$; E_0, E_1 are identity matrices of order Mn and Kn; B^1, S^1, B^2, S^2, $U(\cdot)$, $V(\cdot)$ are matrices of dimension $Mn \times Mn$, $Kn \times Mn$, $Kn \times Kn$, $Mn \times Kn$, $Mn \times Mn$, $Kn \times Kn$ respectively

$$B^i = \begin{Vmatrix} B_1^i \ldots 0 \\ \cdots \\ 0 \ldots B_{n(i)}^i \end{Vmatrix}, \quad S^i = \begin{Vmatrix} S_{11}^i \ldots S_{1n(i)}^i \\ \cdots \\ S_{m(i)1}^i \ldots S_{m(i)n(i)}^i \end{Vmatrix}$$

$$U(\cdot) = \begin{Vmatrix} U_1(\cdot) \ldots 0 \\ \cdots \\ 0 \ldots U_M(\cdot) \end{Vmatrix}, \quad V(\cdot) = \begin{Vmatrix} V_1(\cdot) \ldots 0 \\ \cdots \\ 0 \ldots V_K(\cdot) \end{Vmatrix}$$

$$i = 1, 2;\ n(1) = m(2) = M,\ n(2) = m(1) = K$$

Definition 1. A pure strategy $u_i(\cdot)$ of the object P_i in group A is the function $u_i(\cdot)$: $R_+ \times R^{n(K+M)} \to [0, 1]^n$, i.e. $u_i(t, z) \in [0, 1]^n$ for $(t, z) \in R_+ \times R^{n(K+M)}$ $(i = 1, \ldots, M)$.

In group B, the object $L_j (j = 1, \ldots, K)$ chooses a realization of the strategy $v_j(t) \in [0, 1]^n$ using any available information.

Assume that the initial position $(0, z(0))$, the strategy of group A $u(\cdot) = (u_1(\cdot), \ldots, u_M(\cdot))$, and some realization of the strategy of group B $v(t) = (v_1(t), \ldots, v_K(t))$ are known. Consider the partition Δ of the half-line $0 \leq t < +\infty$ by the system of half-open intervals

$$\tau_i \leqslant t < \tau_{i+1}, \ \Delta_i = \tau_{i+1} - \tau_i \ (i = 0, 1, \ldots, \tau_0 = 0)$$

The discrete analogue of the system of differential equations is written as

$$\begin{aligned} x(\tau_{k+1}) &= x(\tau_k) + \Delta_k \{-x(\tau_k) + (E_0 - U(\tau_k, z(\tau_k)))B^1 x(\tau_k) - \\ &\quad - S^2 V(\tau_k)\, y(\tau_k)\} \\ y(\tau_{k+1}) &= y(\tau_k) + \Delta_k \{-y(\tau_k) + (E_1 - V(\tau_k))\, B^2 y(\tau_k) - \\ &\quad - S^1 V(\tau_k, z(\tau_k))\, B^1 x(\tau_k)\} \end{aligned} \tag{2.1}$$

We use the standard technique to determine Euler's polygonal lines [3, 5] and the notions $z[t]$ generated by the strategy $u(\cdot)$ from the position $(0, z(0))$.

Definition 2. The optimality set of group A in time T $(0 < T < +\infty)$ is the set

$$W_1^T = \{z(0) : z(0) \in \operatorname{int} R_+^{n(K+M)} : [\exists u^{op}(\cdot) : \forall z[t, 0, z(0), u^{op}(\cdot)]$$

$$\exists s : 0 \leqslant s \leqslant T : z(s) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} T_j), \quad x(\tau_0) \in \operatorname{int} R_+^{Mn},$$

$$0 \leqslant \tau_0 \leqslant s]; \ \forall \tau_1 : 0 < \tau_1 < T \ \rceil[\exists u(\cdot) : \forall z[t, 0, z(0), u(\cdot)]$$

$$\exists s : 0 \leqslant s \leqslant \tau_1 \ z(s) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} T_j), \quad x(\tau_0) \in \operatorname{int} R_+^{nM}, \ 0 \leqslant \tau_0 \leqslant s]\}$$

In other words, this is the set of initial states such that, for a game starting from any of these states, group A has a strategy $u^{op}(\cdot)$ which will ensure satisfaction of condition (1.1) not later than at time T, regardless of the actions of group B. This property does not hold for $t < T$.

The strategy $u^{op}(\cdot)$ is called an optimal strategy of group A in time T.

Definition 3. The optimality set of group A in the differential game is the set

$$W_1^* = \bigcup_{0 < T < +\infty} W_1^T$$

Problem 1. For any $T \in (0, +\infty)$ construct the set W_1^T, find the optimal strategy $u^{op}(\cdot)$ of group A, and also construct the set W_1^*.

We similarly formulate and solve Problem 2, which requires constructing the optimality sets and finding the optimal strategy of group B.

3. MULTISTEP GAME OF QUALITY

The solution of Problems 1 and 2 relies on multistep games that approximate the differential game. We accordingly consider the partition Δ of the half-line $[0, +\infty)$ and assume that the motion of the groups of conflict-controlled objects is defined by system (2.1). The objects of group A apply the strategies $u_i(\cdot)$: $T_\Delta \times R^{n(K+M)} \to [0, 1]^n$, $u_i(t, z) \in [0, 1]^n$, $(t, z) \in T_\Delta \times R^{n(K+M)} T_\Delta = \{\tau_0, \tau_1, \ldots\}$, $i = 1, \ldots, M$.

The objects of group B choose $v_j(t) \in [0, 1]^n$, $j = 1, \ldots, K$, using any available information. The termination of this multistep game is determined by the conditions

$$z(t) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} F_j) \tag{3.1}$$

$$z(t) \in (\bigcap_{j=1}^{Kn} T_j^*) \cap (\bigcup_{i=1}^{Mn} C_i) \tag{3.2}$$

$$z(t) \in (\bigcup_{i=1}^{Mn} C_i) \cap (\bigcup_{j=1}^{Kn} F_j) \tag{3.3}$$

$$C_i = \{z:\ z \in R^{n(K+M)},\ x^i \leqslant 0\},\ F_j = \{z : z \in R^{n(K+M)},\ y^j \leqslant 0\}$$

Definition 4. The optimality set of group A in step m is the set

$$W_1{}^m = \{z(0) : z(0) \in R^{n(K+M)} : [\exists u^{op}(\cdot)\, \forall v(t)\, \exists s \in T_\Delta : \tau_0 \leqslant s \leqslant \tau_m : z(s) \in$$
$$\in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} F_j),\ x(\tau) \in \operatorname{int} R_+^{nM},\ \tau_0 \leqslant \tau \leqslant s],\quad \forall \tau_* : \tau_0 \leqslant \tau_* \leqslant \tau_{m-1},$$
$$m \geqslant 1,\ \neg[\exists u(\cdot)\, \forall v(t)\, \exists s \in T_\Delta : \tau_0 \leqslant s \leqslant \tau_*\ z(s) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} F_j),\ x(\tau) \in$$
$$\in \operatorname{int} R_+^{nM},\ \tau_0 \leqslant \tau \leqslant s]\}$$

The strategy $u^{op}(\cdot)$ is called the optimal strategy of group A in step m.

Definition 5. The optimality set of group A in the multistep game is the set

$$W_1^{**} = \bigcup_{T=1}^{\infty} W_1{}^T$$

*Problem 1**. For any $m = 1, \ldots,$ construct the set $W_1{}^m$, find the optimal strategy $u^{op}(\cdot)$ of group A in step m, and also construct the set $W_1{}^{**}$.

*Problem 2** is formulated and solved similarly.

4. NOTATION AND PROPOSITIONS REQUIRED TO SOLVE PROBLEMS 1 AND 1*

Let $\Delta t > 0$,. We take

$$\gamma_0 = \frac{1}{\Delta t} - 1,\quad \Gamma_0 = \gamma_0 E_0,\quad \Gamma_1 = \gamma_0 E_1,\quad x^+ = \begin{cases} x, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$
$$D_0 = S^1 B^1,\ D_m = D_{m-1}\Gamma_0 + D_{m-1}B^1 + (Q_{m-1}S^1 - D_{m-1})^+ B^1$$
$$Q_0 = \Gamma_1 + B^2,\ Q_m = Q_{m-1}\Gamma_1 + Q_{m-1}B^2 + (D_{m-1}S^2 - Q_{m-1})^+ B^2$$

$$R_0 = \Gamma_0,\ R_m = R_{m-1}\Gamma_0 + T_{m-1}S^1B^1$$
$$T_0 = S^2B^2,\ T_m = T_{m-1}\Gamma_1 + T_{m-1}B^2 + (R_{m-1}S^2 - T_{m-1})^+B^2,$$
$$m = 1, \ldots$$

$$q_m^{ij} = \begin{cases} \dfrac{(Q_mS^1)_{ij}}{(D_m)_{ij}}, & \text{if } (D_m)_{ij} > 0; \\ 1, & \text{if } (D_m)_{ij} = 0, (Q_mS^1)_{ij} = 0; \\ +\infty, & \text{if } (D_m)_{ij} = 0, (Q_mS^1)_{ij} > 0; \end{cases}$$

$$\delta_m^{ij} = \begin{cases} \dfrac{(Q_m)_{ij}}{(D_mS^2)_{ij}}, & \text{if } (D_mS^2)_{ij} > 0 \\ 1, & \text{if } (D_mS^2)_{ij} = 0, (Q_m)_{ij} = 0 \\ +\infty, & \text{if } (D_mS^2)_{ij} = 0, (Q_m)_{ij} > 0 \end{cases}$$

$$\theta_m^{ij} = \begin{cases} \dfrac{(T_m)_{ij}}{(R_mS^2)_{ij}}, & \text{if } (R_mS^2)_{ij} > 0 \\ 1, & \text{if } (R_mS^2)_{ij} = 0, (T_m)_{ij} = 0 \\ +\infty, & \text{if } (R_mS^2)_{ij} = 0, (T_m)_{ij} > 0 \end{cases}$$

The following lemmas hold

Lemma 1. For $(B^2S^1)_{ij} \geq (S^1B^1)_{ij}$, $(B^2)_{ij} \geq (S^1B^1S^2)_{ij}$, $i = 1, \ldots, Kn$; $j = 1, \ldots, Mn$ we have $q_m^{ij} \geq 1$, $\delta_m^{ij} \geq 1$ $\forall m = 1, \ldots$.

Lemma 2. For $(B^2)_{ij} \geq (S^1B^1S^2)_{ij}$, $i = 1, \ldots, Kn$; $j = 1, \ldots, Mn$; $\forall (i,j) \exists N_0^i: \forall m \geq N_0^{ij}$ $\theta_m^{ij} \geq 1$.

5. SOLUTION OF MULTISTEP GAME OF QUALITY

The solution procedure applies the dynamic programming principle and develops a discrete backward construction. The method generates the sets W_1^m in a stepwise manner, starting with $m = 1, \ldots$. In the first stage, we find the initial states $z(0)$ that belong to W_1^1, in the second stage we find the states $z(0) \in W_1^2$, i.e. states such that the realization of the optimal group A strategy and the worst realization of group B strategy (for group A) take the system from these $z(0)$ to state $z(1) \in W_1^1$, and so on.

Let us construct the sets W_1^m. We see that

$$W_1^1 = \bigcup_{i=1}^{Kn} (W_1^1)_i,$$

where $(W_1^1)_i = \{z(0): z(0) \in \operatorname{int} R_+^{n(K+M)}: \exists u^{op}(\cdot)\ \forall v(\cdot)\ x(\tau_1) \in \operatorname{int} R_+^{Mn}, (y(\tau_1))_i \leq 0\}$.

Clearly $x(\tau_1) \in \operatorname{int} R_+^{Mn}$ for any realization of group B strategy in two cases:

(a) $(E_0 - \Delta_0E_0)\, x(0) - \Delta_0S^2B^2y(0) \in \operatorname{int} R_+^{Mn}$

(b) $(E_0 - \Delta_0E_0)\, x(0) + \Delta_0 [B^1x(0) - S^2B^2y(0)] \in \operatorname{int} R_+^{Mn}$

$\exists i_0 \in \{1, \ldots, Mn\}$: $\{(E_0 - \Delta_0E_0)\, x(0) - \Delta_0S^2B^2y(0)\}\ i_0 \leqslant 0$

Consider case (a). Substitute into the inequality $(y(\tau_1))_i \leq 0$ the optimal realization of group A strategy $u^{op}(0) = (1, \ldots, 1)$ and the worst (for group A) realization of group B strategy $v(0) = (0, \ldots, 0)$. Then

$$(W_1^1)_i = \Big\{z(0) : z(0) \in \text{int } R_+^{n(K+M)},\ (E_0 - \Delta_0 E_0)\, x(0) -$$
$$- \Delta_0 S^2 B^2 y(0) \in \text{int } R_+^{Mn},\ (S^1 B^1 x(0))_i \geqslant \Big(\Big(\Big(\frac{1}{\Delta_0} - 1\Big) E_1 + B^2\Big) y(0)\Big)_i\Big\}.$$

For $(S^1B^1)_{ik} = 0\ \forall k = 1, \ldots, Mn,\ (W_1^1)_i = \varnothing;\ W_1^1 = \bigcup_{i=1}^{Kn} (W_1^1)_i$

The optimal group A strategy is one of the function $u_i^{op}(\cdot)$: $u_i^{op}(0, z(0)) = (u_i^{op,1}(0, z(0)), \ldots, u_i^{op,Mn}(0, z(0)))$, where

$$u_i^{op,j}(0, z(0)) = \begin{cases} 1, & z(0) \in (W_1^1) \\ \text{Undefined otherwise}; & j = 1, \ldots, Mn \end{cases}$$

When constructing the sets W_1^m for $m \geqslant 2$, we assume that $\Delta_k = \tau_{k+1} - \tau_k = \Delta t > 0\ \forall k = 0, 1, \ldots$. Denote $z(\tau_m)$, $x(\tau_m)$ and $y(\tau_m)$ by z_m, x_m and y_m. From the definition of the sets W_1^m it follows that two conditions are fundamental for their construction

$$\exists i : 1 \leqslant i \leqslant Kn\ (y_m)_i \leqslant 0 \qquad (5.1)$$

$$x_m \in \text{int } R_+^{Mn} \qquad (5.2)$$

Arguing as for $m = 1$, we obtain that if $z_0 \in R_+^{(K+M)n}$

$$\{D_m x_0\}_i \geqslant \{Q_m y_0\}_i$$

then group A will satisfy conditions (5.1) in step m regardless of the action of group B. The procedure to find the states z_0 satisfying (5.1) shows that the realization of the optimal group A strategy and the realization of the optimal group B counter-move depend on the relationship of the matrix elements defining the system dynamics.

Suppose that the following matrix inequalities hold:

$$B^2 S^1 \geqslant S^1 B^1,\ B^2 \geqslant S^1 B^1 S^2$$

Then, by Lemma 1, the realization of the optimal group A strategy is $U^{op}(m) = E_0$ and the optimal counter-move of group B is $V^*(m) = 0$ (matrix zero) $\forall m = 1, \ldots$. We seek $z_0 \in \text{int} R_+^{(K+M)n}$ satisfying (5.2) under the assumption that group A applies the realization $U^{op}(m) = E_0\ \forall m = 1, \ldots$. Under this assumption, we similarly show that (5.2) for any $m = 1, \ldots$ is equivalent to the condition

$$R_m x_0 > T_m y_0$$

In each of the Mn inequalities that define the collection of states z_0 satisfying (5.2), group B chooses $V^*(0)$ in the following way: $(V^*(0))_{jj} = 1$ for $(R_m S^2)_{ij} > (T_m)_{ij}$; $(V^*(0))_{jj} = 0$ for $(R_m S^2)_{ij} \geqslant (T_m)_{ij}$, $j = 1, \ldots, Kn$ (in the ith inequality). By Lemma 2, $\exists N_0^{ij}$: $\forall m \geqslant N_0^{ij}$ we have $(V^*)(k))_{jj} = 0$, $k = 0, \ldots, m - N_0^{ij}$, $(V^*(k))_{jj} = 1$, $k = m - N_0^{ij} + 1, \ldots, m$; $i = 1, \ldots, Mn$; $j = 1, \ldots, Kn$. Thus, for $B^2 S^1 \geqslant S^1 B^1$, $B^2 \geqslant S^2 B^1 S^2$, $S^1 B^1 \neq 0$ in case (a) we obtain

$$W_1{}^m = \bigcup_{i=1}^{Kn} \{z_0 : z_0 \in \text{int } R_+^{(K+M)n}, \forall j : 1 \leqslant j \leqslant m, R_j x_0 -$$

$$- T_j y_0 \in \text{int } R_+^{Mn}, (D_m x_0)_i \geqslant (Q_m y_0)_i\} / \bigcup_{j=1}^{m-1} W_1{}^j$$

$$W_1{}^1 = \bigcup_{i=1}^{Kn} \{z_0 : z_0 \in \text{int } R_+^{(K+M)n}, R_0 x_0 - T_0 y_0 \in \text{int } R_+^{Mn}, (D_0 x_0)_i \geqslant$$

$$\geqslant (Q_0 y_0)_i\}, W_1^{**} = \bigcup_{m=1}^{\infty} W_1{}^m$$

The optimal strategy of group A with this relationship between the parameters is the function $u^{op}(\cdot)$: $R_+{}^{(K+M)n} \to [0, 1]^{Mn}$, i.e. $u^{op}(z) = (u^{op,1}(z), \ldots, u^{op,Mn}(z))$, where

$$u^{op,i}(z) = \begin{cases} 1, & z \in W_1{}^{**} \\ \text{Undefined otherwise} \end{cases}$$

The sets $W_1{}^m$, $W_1{}^{**}$ and the optimal group A strategy $u^{op}(\cdot)$ are determined in exactly the same way both for nonuniform partition of the half-line $[0, +\infty)$ in case (a) and for any partition of the half-line $[0, +\infty)$ in case (b).

Note that the solution of multistep games for case (b) cannot be used to find the sets $W_1{}^T$, $W_1{}^*$ and the optimal strategy $u^{op}(\cdot)$ in the differential game, because such multistep games do not approximate the original differential game. Indeed, in the differential game the initial state $z(0) \in \text{int } R_+{}^{n(K+M)}$, and so $\exists \Delta^* > 0$ such that, for $\Delta_0 < \Delta^*$, $z(0)$ is not the initial state of any multistep game whose initial states satisfy condition (b).

6. SOLUTION OF THE DIFFERENTIAL GAME OF QUALITY

Let $\delta > 0$. Denote by $\{\Delta_\alpha\}_{\alpha \in A_\delta}$ the family of partitions of the half-line $[0, +\infty)$ with the property

$$\sup_k (\tau_{k+1}^{\Delta\alpha} - \tau_k^{\Delta\alpha}) \leqslant \delta$$

and by

$$\{\Delta_\alpha\}_{\alpha \in A_\delta^T}$$

the family of partitions of the interval $[0. T]$, $T > 0$, with the property

$$\sup_k (\tau_{k+1}^{\Delta\alpha} - \tau_k^{\Delta\alpha}) \leqslant \delta$$

In any multistep game generated by the partition $\Delta_\alpha (\beta \in A_\delta)$ of the half-line $[0, +\infty)$ whose solution was obtained in case (a), find the set

$$(W_1^{**})_{\Delta_\alpha} = \bigcup_{n=1}^{\infty} W_1{}^n$$

Similarly in the multistep game generated by the partition $\Delta_\alpha (\alpha \in A_\delta{}^T)$ of the interval $[0, T]$ whose solution was obtained in case (a) find the set

$$(\overline{W}_1)_{\Delta_\alpha} = \bigcup_{n=1}^{N(T)} W_1{}^n$$

where $N(t)$ is the number of steps in this multistep game. Let

$$V_1{}^T = \bigcup_{\delta_0>0} \bigcap_{0<\delta\leq\delta_0} \bigcap_{\{\Delta_\alpha\}} \bigcap_{\alpha\in A_\delta^T} (\overline{W}_1)_{\Delta_\alpha}; \quad V_1 = \bigcup_{\delta_0>0} \bigcap_{0<\delta\leq\delta_0} \bigcap_{\{\Delta_\alpha\}} \bigcap_{\alpha\in A_\delta} (W_1^{**})_{\Delta_\alpha}$$

Theorem 1. Problem 1 has a solution for the following relationships between the game parameters:

1) $B^2S^1 \geqslant S^1B^1,\ B^2 \geqslant S^1B^1S^2,\ S^1B^1 \neq 0;$
2) $B^2S^1 \leqslant S^1B^1,\ S^1B^1S^2 \geqslant B^2,\ S^1S^2 \leqslant E_1,\ S^2S^1 \leqslant E_0,\ S^1B^1 \neq 0;$
3) $B^1 \geqslant S^2B^2S^1,\ S^1S^2 \geqslant E_1,\ S^1B^1 \neq 0;$
4) $B^2S^1 \geqslant S^1B^1,\ B^2 \leqslant S^1B^1S^2,\ S^1S^2 \geqslant E_1,\ S^2S^1 \geqslant E_0,\ S^1B^1 \neq 0$

and $W_1{}^T = V_1{}^T - \bigcup_{0<t<T} V_1{}^t,\ W_1 = V_1.$

In case 1 the proof of Theorem 1 is easily obtained from Lemmas 1 and 2 and the definition of the sets $V_1{}^T$, V_1. In other cases, it is obtained in a similar way.

7. UNCOORDINATED CHOICE OF STRATEGIES BY THE OBJECTS IN A GROUP

In this case, when the object P_i in group A constructs its optimality set, it treats its group A partners as objects from group B. We define as before the pure strategy of object P_i in the group, Euler's polygonal lines, and the motions generated by strategy $u_i(\cdot)$ from the position $(0, z(0))$. Only the definition of the optimality set of group A changes.

*Definition 1**. The optimality set of group A in time $T(0<T<+\infty)$ is the set

$$\overline{W}_1{}^T = \bigcup_{i=1}^{M} (\overline{W}_1{}^T)_i$$

Here

$$(\overline{W}_1{}^T)_i = \{z(0) : z(0) \in \operatorname{int} R_+^{n(K+M)} : [\exists u_i^{op}(\cdot) : \forall z[t, 0, z(0), u_i^{op}(\cdot)]\, \exists s : 0 \leqslant s \leqslant T : z(s) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} T_j), x(\tau_0) \in \operatorname{int} R_+^{nM}, 0 \leqslant \tau_0 \leqslant s], \forall \tau_1 : 0 < \tau_1 < T \;\neg [\exists u_i(\cdot) : \forall z[t, 0, z(0), u_i(\cdot)]\, \exists s : 0 \leqslant s \leqslant \tau_1\, z(s) \in (\bigcap_{i=1}^{Mn} S_i) \cap (\bigcup_{j=1}^{Kn} T_j),\ x(\tau_0) \in \operatorname{int} R_+^{nM},\ 0 \leqslant \tau_0 \leqslant s]\}$$

The strategy $u_i^{op}(\cdot)$ is the optimal strategy of object P_i in the group.

*Definition 2**. The optimality set of group A in the differential game is the set

$$\overline{W}_1 = \bigcup_{0<T<+\infty} \overline{W}_1{}^T$$

The optimality set $\bar{W}_1$ is also representable in the form

$$\overline{W}_1 = \bigcup_{i=1}^{M} (\overline{W}_1)_i \quad (\overline{W}_1)_i = \bigcup_{0<T<+\infty} (\overline{W}_1{}^T)_i$$

Here $(W_1)_i$ is called the optimality set of object P_i in a group in the differential game.

Problems 1 and 2 are formulated similarly. In the same way, we consider multistep games of quality that approximate the differential game with uncoordinated choice of strategies by the objects in a group.

Let us introduce some notation. Let A be an $m\times n$ matrix. then $A^*_{[k,l]}(1\leqslant k\leqslant l\leqslant n)$ is a matrix of the following form: $a_{ij}{}^* = -a_{ij}{}^-$ for $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k-1$; $l+1\leqslant j\leqslant n$; $a_{ij}{}^* = a_{ij}{}^+$ for $1\leqslant i\leqslant m$, $k\leqslant j\leqslant l$; and $\bar{A}_{[k,l]}$ is the matrix of the following form: $\bar{a}_{ij} = -a_{ij}{}^-$ for $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k-1$; $l+1\leqslant j\leqslant n$; $\bar{a}_{ij} = a_{ij}$ for $1\leqslant i\leqslant m$, $k\leqslant j\leqslant l$. Here

$$x^- = \begin{cases} 0, & x\geqslant 0 \\ -x, & x<0 \end{cases}$$

If i_0: $1\leqslant i_0\leqslant M$, then

$$D_0 = S^{1,*}_{[(i_0-1)n+1,\, i_0 n]}\cdot B^1, \quad D_m = D_{m-1}\Gamma_0 + D_{m-1}B^1 + (Q_{m-1}S^1 - D_{m-1})^*_{[(i_0-1)n+1,\, i_0 n]}\cdot B^1$$

$$Q_0 = \Gamma_1 + B^2, \quad Q_m = Q_{m-1}\Gamma_1 + Q_{m-1}B^2 + (D_{m-1}S^2 - Q_{m-1})^+ B^2$$

$$R_0 = \Gamma_0, \quad R_m = R_{m-1}\Gamma_0 + R_{m-1}B^1 + \overline{(T_{m-1}S^1 - R_{m-1})}_{[(i_0-1)n+1,\, i_0 n]}B^1$$

$$T_0 = S^2B^2, \quad T_m = T_{m-1}\Gamma_0 + T_{m-1}B^2 + (R_{m-1}S^2 - T_{m-1})^+ B^2$$

Since in this case the basis for the solution of the differential game is analysis of multistep games approximating the differential game, we will show that the quantities introduced above make it possible to write the optimality set in the multistep game of group A with uncoordinated choice of strategies by the objects in the group. Let $B^2S^1\geqslant S^1B^1$, $B^2\geqslant S^1B^1S^2$. We take $m = 1$.

Consider the construction of the sets $(\bar{W}_1{}^m)_{i_0}$ $(1\leqslant i_0\leqslant M)$. Then from the condition $(y(\tau_1))_i\leqslant 0$ with uncoordinated choice we obtain

$$[S^{1,*}_{[(i_0-1)n+1,\, i_0 n]}\cdot B^1 x(0) \geqslant (\Gamma_1 + B^2)\, y(0)]_i, \quad i = 1, \ldots, Kn$$

Combined with condition (a), this gives

$$(\overline{W}_1{}^1)_{i_0} = \bigcup_{i=1}^{Kn} \{z(0) : z(0) \in \operatorname{int} R_+^{n(K+M)},\ \Gamma_0 x(0) - S^2B^2 y(0) \in$$
$$\in \operatorname{int} R_+^{Mn}\ [S^{1,*}_{[(i_0-1)n+1,\, i_0 n]}B^1 x(0) \geqslant (\Gamma_1 + B^2)\, y(0)]_i\}$$

For

$$(S^{1,*}_{[(i_0-1)n+1,\, i_0 n]}\cdot B^1)_{ik} = 0\ \forall i = 1, \ldots, Kn;\ k = 1, \ldots, Mn\ (\overline{W}_1{}^1)_{i_0} = \varnothing$$

$$(\overline{W}_1{}^m)_{i_0} \quad (m = 2, \ldots):$$

We similarly find the sets $(\bar{W}_1{}^m)_{i_0}$ $(m = 2, \ldots)$:

$$(\overline{W}_1{}^m)_{i_0} = \bigcup_{i=1}^{Kn} \{z(0) : z(0) \in \operatorname{int} R_+^{(K+M)n},\ \forall j : 1\leqslant j\leqslant m,\ R_j x(0) - T_j y(0) \in$$

$$\in \operatorname{int} R_+^{Mn}, (D_m x(0))_i \geqslant (Q_m y(0))_i\} \setminus \bigcup_{j=1}^{m-1} (\overline{W}_1{}^j)_{i_0}, \overline{W}_1 = \bigcup_{m=1}^{\infty} \bigcup_{i_0=1}^{m} (\overline{W}_1{}^m)_{i_0}$$

As before, the optimal strategies of the objects in a group are written out for this relationship between the matrix elements. Theorem 1 holds for the differential game.

Note that with uncoordinated choice of the strategies by the objects in a group, the optimality sets $\bar{W}_1{}^T$, $\bar{W}_1$ are "smaller" than the optimality sets W_1T, $W_1{}^*$, i.e. $\bar{W}_1{}^T \subset W_1{}^T$, $W_1 \subseteq W_1{}^*$, which follows from their definition.

8. EXAMPLE

We will show, for instance, that $W_1 \subset W_1{}^*$. Consider the differential game of quality of a group consisting of two objects and a group consisting of one object. The motion of the objects of the first group is described by the system

$$x_1{}^{\cdot}(t) = -x_1(t) + (1 - u_1(t))\, x_1(t) - v(t) \cdot 2y(t)$$
$$x_2{}^{\cdot}(t) = -x_2(t) + (1 - u_2(t))\, x_2(t) - v(t) \cdot 2y(t)$$

and the motion of the second group is described by the equation

$$y^{\cdot}(t) = -y(t) + (1 - v(t))\, 2y(t) - u_1(t)\, x_1(t) - u_2(t)\, x_2(t)$$

In this game, condition (1) of Theorem 1 is satisfied. In the case of a coordinated choice of strategies by the objects in the group, we obtain

$$W_1{}^* = \{(x_1(0),\ x_2(0),\ y(0)):\ (x_1(0),\ x_2(0),\ y(0)) \in \operatorname{int} R_+{}^3,\ y(0) < \sqrt{x_1(0)\, x_2(0)}\}$$

$$u_i^{op}(\cdot) : u_i^{op}(x_1(0), x_2(0), y(0)) = \begin{cases} 1, & (x_1(0), x_2(0), y(0)) \in W_1{}^*, \\ \text{Otherwise undefined} \end{cases}$$

With uncoordinated choice of strategies by the objects in the group, we obtain the optimality sets $(\bar{W}_1)_1$, $(\bar{W}_1)_2$ of the first and the second object in the group:

$$(\overline{W}_1)_1 = \{(x_1(0),\ x_2(0),\ y(0)):\ (x_1(0),\ x_2(0),\ y(0)) \in \operatorname{int} R_+{}^3,\ y(0) < \sqrt{x_1(0)\, x_2(0)},\ y(0) < {}^1\!/_2 x_1(0)\},$$

$$(\overline{W}_1)_2 = \{(x_1(0),\ x_2(0),\ y(0)):\ (x_1(0),\ x_2(0),\ y(0)) \in \operatorname{int} R_+{}^3,\ y(0) < \sqrt{x_1(0)\, x_2(0)},\ y(0) < \frac{1}{2} x_2(0)\}$$

The optimality set of the first group is

$$\overline{W}_1 = (\overline{W}_1)_1 \cup (\overline{W}_1)_2$$

The optimal strategies in this case are

$$u_i^{op}(\cdot) : u_i^{op}(x_1(0), x_2(0), y(0)) = \begin{cases} 1, & (x_1(0), x_2(0), y(0)) \in (\overline{W}_1)_i \\ \text{Undefined otherwise;} & i = 1, 2 \end{cases}$$

Thus, $\bar{W}_1 \subset W_1{}^*$. If the initial states of the objects in the groups are $(x_1(0), x_2(0), y(0)) = (1, 1, 0.75)$, the first group of objects with coordinated choice of strategies by the objects in the group achieves condition (1.1) not later than in a time $t^* = (\ln 7 - \ln 4)/4$ applying its optimal strategies $u_1{}^{op}(\cdot)$, $u_2{}^{op}(\cdot)$. If the objects in the first group act without coordination, they cannot achieve condition (1.1) for any t with these initial conditions.

9. CONCLUSION

As we have noted before [3], the proposed method is useful for numerical implementation, because it solves approximating multistep games, which are a discrete analogue of the original differential game. This method determines optimal strategies and optimality sets without solving the system of differential equations, whose solution involves considerable computational difficulties when the system is large and the groups contain many objects. Also note that the differential game of the two groups of objects considered in this paper adequately describes the process of conflict interaction of two groups of economic systems functioning in continuous time [6].

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